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Hall Effect in a Superconductor Not in the Vortex State*

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The Hall electric field inside a superconductor not in the vortex state is calculated using the time-dependent Ginzburg-Landau equations. The field exists only in the skin of the superconductor. Part of the electric field penetrating the material dies out in a London penetration length and part dies out in a coherence length. For an applied magnetic field of 1 G, the field is 10^{-7}–10^{-8} V/m in most pure superconductors, and is proportional to the square of the applied magnetic field. The gradient of the electrochemical potential is zero, so there is no Hall voltage. The charge density producing the electric field is a dipole layer, a penetration length thick, at the surface of the material. This means that the contact potential has a part that depends quadratically on the applied magnetic field.

I. INTRODUCTION

Since the work of Onnes and Hof,[1] it has been assumed that there is no Hall effect in a superconductor.[2] Work done since then has also indicated that the Hall effect vanishes. Lewis[3] measured the Hall voltage in a vanadium sample and found an upper bound for it smaller than the normal-state value, from which he supposed that it was probably zero.[4] More recently Bok and Klein[5] have investigated the Hall effect using two methods. Using direct Ohmic contacts to the sample, which was the method used by Lewis,[3] they also found zero voltage. However, using the Kelvin technique they found that the magnitude of the contact potential depended quadratically on the applied magnetic field. This implies that a Hall electric field exists inside a superconducting material, but that the corresponding Hall voltage is canceled by a contact potential.

Naively, one might say that there should not be a Hall effect in a superconductor because of the Meissner effect; magnetic fields are excluded from the superconductor. In fact magnetic fields penetrate typically to a depth of about 10^{-5} cm, the London penetration depth, so that it is not correct to claim that the Meissner effect prohibits a Hall effect.

There is, of course, a Meissner current in the skin of a superconducting material in a magnetic field. The superfluid velocity corresponding to this current is position dependent; it dies out approximately exponentially in a London penetration length. Roughly speaking, the chemical potential of the superconducting electrons consists of two parts. The first part is the contribution that would be found in equilibrium with no external magnetic field. This part is a constant throughout the material. The second is the kinetic energy due to the motion of the electrons needed to screen out the magnetic field. Since the superfluid velocity is position dependent, so is the chemical potential. The electrochemical potential should be constant, so an electric field must exist to cancel the chemical potential gradient. The electric field was predicted by London[6] for a charged superfluid. Lewis[4] suggested that it should also exist in a superconductor, and have the value

$$\vec{E} = (1/e)\vec{\nabla}\tfrac{1}{2}mv^2, \qquad (1)$$

where v is the velocity of the electron superfluid. No Hall voltage was found in the experiments using direct Ohmic contacts mentioned above, because there is no electrochemical potential gradient.

The discussion of the preceding paragraph is not a complete picture of the Hall effect in a superconductor, however. The derivation of (1) is essentially within the framework of the two-fluid model. Except at $T=0$ there is a normal fluid which must also be in equilibrium. The electric field (1) also acts on the normal fluid, so a problem arises in maintaining equilibrium for both the superfluid and the normal fluid.

In this paper time-dependent Ginzburg-Landau theory is used to treat the problem of the Hall field in a pure superconductor. In order to ensure equilibrium, a nonzero, static, electric field exists in the skin of the superconductor. The electric field consists of two parts which decrease as a function of the distance from the surface. One part dies out in a London penetration length and the other in a coherence length. For a magnetic field of 1 G the magnitude of the electric field is 10^{-7}–10^{-8} V/m in most pure materials. The gradient of the electrochemical potential is zero. In addition, the charge density required to produce the Hall field leads, in effect, to a surface dipole layer whose magnitude depends on the square of the applied magnetic field. This is qualitatively (and roughly quantitatively) in agreement with the experimental results of Bok and Klein.[5]

II. GINZBURG-LANDAU EQUATIONS

In the theory of superconductivity, the Ginzburg-Landau theory has had great success in treating a variety of problems. As originally proposed,[7,8] the Ginzburg-Landau equations describe the equilibrium properties of a superconductor by relating spatial variations of the order parameter (the gap function) to the vector potential and the current. Time dependence has since been introduced into the theory in order to treat nonequilibrium situations.[9-12] The resulting set of equations relate the variations of the order parameter in time and space to the vector potential, the current, the scalar potential, and the charge density.

The Hall effect in a superconductor is not a time-dependent effect, but time-dependent Ginzburg-Landau theory must be used to treat the problem since the discussion of the Hall effect deals with an electric field and related charge densities. There is no provision in ordinary, non-time-dependent Ginzburg-Landau theory for the introduction of a static electric field.

The driving term for the Hall effect is the kinetic energy of the superelectrons due to current flow in the superconductor, i.e., the kinetic energy of the moving Cooper pairs. Abrahams and Tsuneto[9] have considered this kinetic energy and have treated it carefully in their derivation of the time-dependent Ginzburg-Landau theory. The net result is a local shift in the chemical potential in the Ginzburg-Landau equations due to the current flow.

In the Gor'kov version of the BCS theory of superconductivity,[13] the "anomalous" Green's function $F \sim \langle\psi\psi'\rangle$ (ψ destroys an electron) and the energy-gap function Δ vary as $e^{-2i\mu t/\hbar}$, where μ is the chemical potential. This time behavior was omitted from the Gor'kov equations of motion for the superconducting Green's function and the anomalous Green's function as originally derived.[13] The time-dependent Ginzburg-Landau equations for the energy-gap function are derived from these Gor'kov equations,[9-12] so they do not include the oscillatory time behavior either. To restore this time dependence, the time derivatives appearing in the Ginzburg-Landau equations as derived from the original Gor'kov equations[13] must be modified through

$$\frac{\partial}{\partial t}\,\Delta \to \left(\frac{\partial}{\partial t} + 2i\mu\right)\Delta\,, \qquad \frac{\partial}{\partial t}\,\Delta^\dagger \to \left(\frac{\partial}{\partial t} - 2i\mu\right)\Delta^\dagger\,. \tag{2}$$

As Abrahams and Tsuneto[9] have pointed out, the extra kinetic-energy term should be included in the chemical potential when the substitution defined in (2) is made. The kinetic energy is due to the motion of a Cooper pair, so the extra term is $P^2_{\text{pair}}/2M_{\text{pair}}$:

$$P^2_{\text{pair}} = \hbar[-i\vec{\nabla} + (2e/\hbar c)\vec{A}]^2\,.$$

Thus, the substitution (including the shift in chemical potential due to the kinetic energy of the moving Cooper pairs) made for $\partial/\partial t$ has the following form:

$$\frac{\partial}{\partial t}\,\Delta \to \left\{\frac{\partial}{\partial t} + 2i\left[\mu + \frac{\hbar^2}{8m}\left(-i\vec{\nabla} + \frac{2e}{\hbar c}\,\vec{A}\right)^2\right]\right\}\Delta\,, \qquad \frac{\partial}{\partial t}\,\Delta^\dagger \to \left\{\frac{\partial}{\partial t} - 2i\left[\mu + \frac{\hbar^2}{8m}\left(+i\vec{\nabla} + \frac{2e}{\hbar c}\,\vec{A}\right)^2\right]\right\}\Delta^\dagger\,. \tag{3}$$

Although a differential equation for the order parameter can be written down at zero temperature, the real utility of the Ginzburg-Landau approach is when the order parameter is small, i.e., when the temperature is close to, but slightly below, the critical temperature. For $1 - T/T_c \ll 1$ the form of the time-dependent Ginzburg-Landau equation used in this analysis has been derived essentially by Abrahams and Tsuneto.[9]

As they have discussed, the coefficient of the second-order time derivative is much smaller than that for the first order, so only the first-order time derivative is kept here:

$$\left[a + b\,|\Delta|^2 + c\left\{\hbar\,\frac{\partial}{\partial t} + 2i\left[\mu - e\phi + \frac{\hbar^2}{8m}\left(-i\vec{\nabla} + \frac{2e}{\hbar c}\,\vec{A}\right)^2\right]\right\} + d\left(-i\vec{\nabla} + \frac{2e}{\hbar c}\,\vec{A}\right)^2\right]\Delta(\vec{r}, t) = 0\,. \tag{4}$$

The $\Delta(\vec{r}, t)$ is the gap function, and is proportional to the order parameter,

$$a = 1 - T/T_c\,, \quad b = -\tfrac{7}{8}\,\zeta(3)/(\pi k T_c)^2\,,$$

$$c = -\pi/8kT_c\,,$$

$$d = -\frac{(\frac{7}{8})\,\zeta(3)}{(\pi k T_c)^2}\,\frac{\hbar^2 v_F^2}{6} = \frac{\hbar^2 v_F^2}{6}\,b\,.$$

The $\zeta(x)$ is the Riemann ζ function, so that $\zeta(3) \cong 1.202$; T_c is the critical temperature; and v_F is the Fermi velocity. In equilibrium for zero fields, this equation reduces to

$$a + b\,|\Delta_0|^2 = 0\,, \tag{5}$$

yielding the equilibrium gap function.

The equation relating the current and the gap function is the usual expression involving gradients of the order parameter

$$\vec{J} = -i\alpha\left[\Delta^\dagger\left(\vec{\nabla} + \frac{2ie}{\hbar c}\,\vec{A}\right)\Delta - \Delta\left(\overleftarrow{\nabla} - \frac{2ie}{\hbar c}\,\vec{A}\right)\Delta^\dagger\right]\,, \tag{6}$$

where

$$\alpha = -\frac{\frac{7}{16}\zeta(3)\hbar n_0 e}{m(\pi k T_c)^2} = \frac{\hbar n_0 e}{2m} b \quad .$$

In time-dependent Ginzburg-Landau theory, there is also an expression relating the charge density ρ to the gap function:

$$\rho = \rho_n + \rho_s \ ,$$

where ρ_n is the out-of-balance charge density of the normal material, ρ_s is the out-of-balance charge density due to the superconducting state, and these are given by

$$\rho_s = \beta\Big[-\tfrac{1}{3}(\hbar^2/2m)\nabla^2|\Delta|^2 + i\Big(\Delta^\dagger\Big\{\hbar\frac{\partial}{\partial t} + 2i\Big[\mu - e\phi + \frac{\hbar^2}{8m}\Big(-i\vec{\nabla} + \frac{2e}{\hbar c}\vec{\mathrm{A}}\Big)^2\Big]\Big\}\Delta - \Delta\Big\{\hbar\frac{\partial}{\partial t} - 2i\Big[\mu - e\phi + \frac{\hbar^2}{8m}\Big(i\vec{\nabla} + \frac{2e}{\hbar c}\vec{\mathrm{A}}\Big)^2\Big]\Big\}\Delta^\dagger\Big)\Big], \tag{7}$$

$$\rho_n = -\tfrac{3}{2}(n_0 e^2/\epsilon_F)\phi \ , \tag{8}$$

where

$$\beta = \frac{3}{\hbar v_F^2}\alpha = -\frac{21\zeta(3)n_0 e}{16 m v_F^2(\pi k T_c)^2} \quad .$$

The equation for ρ_s has been derived by Schmid,[10] but without the term corresponding to the extra kinetic energy of the pair motion. Abrahams and Tsuneto[9] derived ρ_s with the kinetic-energy term, but they did not include the first term, which arises explicitly from spatial variations of the order parameter. The Ginzburg-Landau equations used here are exactly those derived by Abrahams and Tsuneto except for the first term in (7). Abrahams and Tsuneto confined themselves to first-order derivatives in their derivation of the charge-density equation. The second-order spatial derivative in (7) appears in a natural way[10] and would have appeared in the work by Abrahams and Tsuneto if they had elected to consider such terms. The term is of interest so it is included here.[14]

III. HALL EFFECT

The geometry that will be considered is a semi-infinite material with a plane surface, such that $\hat{x}$ is the interior normal. The exterior magnetic field is along the z direction.

Assume that inside the superconductor the magnetic field dies off exponentially, so that

$$\vec{\mathrm{H}} = \hat{z}H_0 e^{-x/\lambda} \quad \text{and} \quad \vec{\mathrm{A}} = -\hat{y}\lambda H_0 e^{-x/\lambda} \ . \tag{9}$$

All spatial dependence in the problem should be on x alone.

The case to be considered is one in which the applied magnetic field H_0 is small, so that only powers of H_0 up to the second need be kept (H_0^2 appears naturally in the equation because of the presence of A^2). The gap function under these conditions should be very nearly the same as the equilibrium, zero-field value Δ_0, so the following form is assumed:

$$\Delta = \Delta_0[1 + f(x)]e^{-2iKt/\hbar} \ , \tag{10}$$

where Δ_0 is the equilibrium solution of the Ginzburg-Landau equation, given by Eq. (5), K is a constant, and $f(x)$ is a small $[|f(x)| \ll 1]$ real function of position. This form for Δ, and the Ginzburg-Landau equation, yield an equation involving f and K:

$$\{a + b\Delta_0^2[1+f(x)]^2 + 2ic(-K + \mu - e\phi)\} \times \Delta_0[1+f(x)]e^{-2iKt/\hbar} + \Big(d + 2ic\frac{\hbar^2}{8m}\Big) \times \Delta_0 e^{-2iKt/\hbar}\Big(-\frac{d^2 f}{dx^2} + s^2 e^{-2x/\lambda}\Big) = 0 \ ,$$

$$s^2 = 4e^2\lambda^2 H_0^2/\hbar^2 c^2 \quad . \tag{11}$$

Only terms up to second order in H_0 are to be kept. Since f must surely depend on H_0, a term like $s^2 f$ may be ignored. Also, terms of higher order than the first in f can be ignored since it is reasonable to expect that $f \sim H_0^2$.

After cancelling the $e^{-2ikt/\hbar}$, the real and imaginary parts of what is remaining in Eq. (11) should vanish separately, so that

$$2b\Delta_0^2 f(x) + d[-f''(x) + s^2 e^{-2x/\lambda}] = 0 \ , \tag{12}$$

$$(-K + \mu - e\varphi)[1 + f(x)] + (\hbar^2/8m)[-f''(x) + s^2 e^{-2x/\lambda}] = 0 \ . \tag{13}$$

Equation (12) may be solved for $f(x)$. The solution of the homogeneous equation contains a decreasing and an increasing exponential function of position, but the coefficient of the increasing exponential must vanish since the order parameter cannot grow indefinitely. Consequently,

$$f_H = G e^{-(2b/d)^{1/2}\Delta_0 x} \ , \tag{14a}$$

where $(2b/d)^{1/2}\Delta_0 = \kappa/\lambda$, and κ is the famous Ginzburg-Landau parameter.

Since $e^{-2x/\lambda}$ is the only explicit spatial dependence in the equation, the particular solution consists of only one term,

$$f_p = \eta e^{-2x/\lambda} \ , \tag{14b}$$

where

$$\eta = \frac{s^2 d}{4d/\lambda^2 - 2b\Delta_0^2} = \frac{s^2\lambda^2}{4 - \kappa^2} \ .$$

In order to ensure that no current enters or leaves the sample, the normal derivative of the gap function at a boundary must vanish in Ginzburg-Landau theory[7]:

$$\hat{n}\cdot\left(\vec{\nabla}-\frac{2ie}{\hbar c}\vec{A}\right)\Big|_{\text{boundary}}=0\ ,$$

so that

$$\frac{df}{dx}\Big|_{x=0}=0\ ,$$

since the vector potential is parallel to the surface in this case, and, finally,

$$f(x)=\eta\left[e^{-2x/\lambda}-(2/\kappa)e^{-\kappa x/\lambda}\right]\ . \tag{15}$$

It has been required that $|f|\ll 1$. Using expression (15) and the fact that $\kappa=2e\sqrt{2}H_c\lambda^2/\hbar c$, it is possible to examine this requirement more closely:

$$|f(0)|=\frac{\kappa}{2(2+\kappa)}\left(\frac{H_0}{H_c}\right)^2\ll 1\ ,$$

where H_c is the thermodynamic critical field. For type-I materials $\kappa<1/\sqrt{2}$. For such values of κ, $|f|\ll 1$ is satisfied well for all applied fields up to the critical field. For type-II materials $\kappa>1/\sqrt{2}$, but this analysis applied only to the nonvortex state, so $H_0<H_{c1}$, where H_{c1} is the lower critical field. Roughly $H_{c1}=H_c(\ln\kappa)/(2\kappa)$, so for type-II materials $|f|\ll 1$ is also satisfied fairly well for applied fields up to H_{c1}.

The ϕ, μ, and κ can now be determined. If there were no magnetic field present, ϕ would be zero and $K=\mu_0$. With a small magnetic field, μ should be close to μ_0, where μ_0 is the chemical potential deep inside the material, where there are no fields or currents, and its spatial dependence should be similar to that of f. Similarily, ϕ should be small and its spatial dependence should also be similar, so the following forms are assumed:

$$\begin{aligned}\phi&=\tilde{\phi}_1e^{-2x/\lambda}+\tilde{\phi}_2e^{-\kappa x/\lambda}\ ,\\ \mu&=\mu_0+\delta\mu\ ,\\ \delta\mu&=\delta\tilde{\mu}_1e^{-2x/\lambda}+\delta\tilde{\mu}_2e^{-\kappa x/\lambda}\ .\end{aligned} \tag{16}$$

These forms (16) imply that $K=\mu_0$, since deep in the interior Eq. (13) reduces to

$$-K+\mu_0=0\ .$$

With the assumptions (16) above, Eq. (13) has a simple form,

$$\left[\delta\tilde{\mu}_1-e\tilde{\phi}_1-\frac{\hbar^2}{8m}\left(\frac{4}{\lambda^2}\ \eta-s^2\right)\right]e^{-2x/\lambda}+\left(\delta\tilde{\mu}_2-e\tilde{\phi}_2+\frac{\hbar^2}{4m}\ \frac{\kappa\eta}{\lambda^2}\right)e^{-\kappa x/\lambda}=0\ . \tag{17}$$

Equation (15) is a solution of the problem, since from $f(x)$ [i.e., $\Delta(x)$] it is possible to find ρ, the charge density, by using Eqs. (7) and (8). The scalar potential can then be calculated via Poisson's equation along with the boundary condition that deep inside the material, far from the surface, where the magnetic field and the current are zero, the electric field vanishes. From Eq. (17) it is then possible to find $\delta\mu$:

$$\tilde{\phi}_1=\frac{16\pi\beta\hbar^2\Delta_0^2\eta/(3m\lambda^2)}{4/\lambda^2-6\pi n_0e^2/\epsilon_F}\ , \tag{18}$$

$$\tilde{\phi}_2=\frac{8\pi\beta\hbar^2\Delta_0^2\kappa\eta/(3m\lambda^2)}{\kappa^2/\lambda^2-6\pi n_0e^2/\epsilon_F}\ , \tag{19}$$

where

$$6\pi n_0e^2/\epsilon_F\sim 10^{16}\,\text{cm}^{-2},\quad 4/\lambda^2\sim 10^{10}\,\text{cm}^{-2}\ .$$

Since the second term in the denominator of Eqs. (18) and (19) completely dominates the first, the first term may be ignored:

$$\tilde{\phi}_1\cong-8\beta\hbar^2\Delta_0^2\epsilon_F\eta/(9mn_0e^2\lambda^2)\ , \tag{20}$$

$$\tilde{\phi}_2\cong 4\beta\hbar^2\Delta_0^2\epsilon_F\kappa\eta/(9mn_0e^2\lambda^2)\ , \tag{21}$$

$$\delta\tilde{\mu}_1\cong\eta\frac{\hbar^2}{m}\ \frac{1}{\lambda^2}\left(\frac{1}{2}-\frac{8}{9}\ \frac{\beta\Delta_0^2\epsilon_F}{n_0e}\right)-s^2\ \frac{\hbar^2}{8m}\ , \tag{22}$$

$$\delta\tilde{\mu}_2\cong-\kappa\eta\frac{\hbar^2}{m}\ \frac{1}{\lambda^2}\left(\frac{1}{4}-\frac{4}{9}\ \frac{\beta\Delta_0^2\epsilon_F}{n_0e}\right)\ . \tag{23}$$

For a pure material near the critical temperature Δ_0 and λ are given below.[15] The electric field corresponding to the scalar potential in (20) and (21) can be written in terms of more recognizable variables:

$$\begin{aligned}\Delta_0^2&=[8/7\zeta(3)](\pi kT_c)^2(1-T/T_c)\ ,\\ \frac{1}{\lambda^2}&=\left(\frac{8\pi n_0e^2}{mc^2}\right)\left(1-\frac{T}{T_c}\right)\ ,\end{aligned} \tag{24}$$

$$\vec{E}=\hat{x}\ \frac{H_0^2}{8\pi}\ \frac{4/\lambda}{3n_0e(4-\kappa^2)}\ (4e^{-2x/\lambda}-\kappa^2e^{-\kappa x/\lambda})\ . \tag{25}$$

Table I contains typical values of $\vec{E}$ for a few materials.

TABLE I. Typical values for $\vec{E}$ (H_0 in gauss).

Material	$|\vec{E}|/\left[\frac{H_0^2}{8\pi}\left(1-\frac{T}{T_c}\right)^{1/2}\right]$ (in V/m)
Pb	$(8.2\times10^{-4})e^{-2x/\lambda}-(4.1\times10^{-6})e^{-\kappa x/\lambda}$
Al	$(4.8\times10^{-4})e^{-2x/\lambda}-(1.7\times10^{-8})e^{-\kappa x/\lambda}$
Nb	$(5.2\times10^{-4})e^{-2x/\lambda}-(1.3\times10^{-6})e^{-\kappa x/\lambda}$
Sn	$(7.8\times10^{-4})e^{-2x/\lambda}-(9.7\times10^{-7})e^{-\kappa x/\lambda}$
V	$(4.5\times10^{-4})e^{-2x/\lambda}-(1.4\times10^{-8})e^{-\kappa x/\lambda}$

The charge density which produces the Hall electric field (25) is an effective-surface dipole layer (of thickness ~ λ). The corresponding potential difference δV across this layer depends quadratically on the applied magnetic field and is given by

$$\delta V=\frac{H_0^2}{8\pi}\,\frac{1}{n_0e}\,\frac{4}{3(2+\kappa)}\quad . \tag{26}$$

Although there is no voltage across the material because there is no electrochemical potential gradient, δV would appear as a shift in the work function of the material. This is in agreement with the experiment of Bok and Klein,[5] who observed that the contact potential of their superconducting sample had a part which behaved roughly in agreement with (26).

IV. CONCLUSION

This analysis shows that there is a Hall field in superconducting materials not in the mixed state. Unlike the normal Hall effect, there is no Hall voltage. In a superconductor the Hall field arises in order to ensure that there be no electrochemical potential gradient. The absence of a voltage explains why attempts to measure a superconducting Hall voltage in the past have led to a null result.[1,3,5] There is a dipole layer at the surface, however, which appears to have been seen.[5]

The magnitude of the Hall field in the superconducting state is comparable, but not identical, to the magnitude of the Hall field in the normal state. Of course, in the superconducting state the field does not exist in the bulk of the material; it exists only in the skin.

Also, there is no Hall coefficient that can be defined in the usual way. The Hall electric field derived using the time-dependent Ginzburg-Landau equations has two parts: One is an exponentially decreasing function of position that dies out in about a coherence distance and the other is an exponential that dies out in a London penetration length. This means that the spatial behavior of the Hall field is different from the spatial behavior of $\vec{J}\times\vec{H}$, so they cannot be simply proportional to one another.

If Eq. (1), the London result, were correct, the potential difference across the dipole layer would be given by

$$\delta V_{\text{London}}=\frac{H_0^2}{8\pi}\,\frac{1}{n_se}\ , \tag{27}$$

where n_s is the density of superconducting electrons. The δV in (26) is roughly the same order of magnitude as the London result (27), but it differs from the London result in some respects. First, n_0 (the total electronic density) appears rather than n_s, so as $T\rightarrow T_c$ δV does not diverge as δV_{London} does. Second, κ appears in (26), which is a manifestation of the fact that the Hall field depends on the coherence length as well as the penetration length.

The calculation presented here applies to pure materials. In the case of a dirty superconductor, a result similar to (26) should also be found, since the Ginzburg-Landau equations for the dirty case are similar in form to Eqs. (4), (6), and (7). However, the values of some of the constants in the equations take on different values, depending on the value of the electronic mean free path. The net result for the dirty case, however, is essentially a redefinition of the Ginzburg-Landau parameter κ. Thus, in the end an expression like (26) for the change in the work function should exist for dirty materials, too.

The results of this calculation give a qualitative picture of the Hall effect in a superconductor not in the vortex state. The results describe the dependence of the electric field and the work function on the various parameters of the problem (e.g., H_0, κ, and λ). Quantitatively, the results should agree with experiment to within an order of magnitude, since the model of a superconductor used here is simple, and a complicated band structure has been replaced by a free-electron approximation. Also, the calculation is limited by the fact that $T\cong T_c$, but this is a limitation of virtually all calculations using Ginzburg-Landau theory. Even though the results are strictly true only near T_c, qualitatively (25) and (26) probably describe the behavior of the superconductor even well below T_c.

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Electron Spin Waves in Nonmagnetic Conductors: Self-Consistent-Field Theory*

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A general derivation of the electrodynamic response of a quantum many-electron gas in a nonmagnetic conducting solid immersed in an applied magnetic field is given. Self-consistent-field (SCF) theory of the equation of motion of the one-electron density matrix is used in such a way as to include, from the outset, one-electron effects such as complex energy band structure, spin-orbit coupling, and spin paramagnetism. This treatment specifically omits exchange effects such as those encountered in an extended random-phase approximation or Landau-Fermi liquid theory. The aim is to study the properties of wave propagation in the gas, looking for spin waves and/or characteristic effects which uniquely involve the spin degree of freedom and the paramagnetism of the equilibrium state. The derived results contain terms which have been neglected previously and terms which do not evolve from a simple generalization of previous treatments of the quantum dielectric theory of a Fermi gas. There are interesting spin effects in the plasma wave properties both *with and without* spin-orbit mixing of the one-electron states. In an effective mass approximation for the one-electron states, it is shown that there are resonances and cutoffs associated with electron spin resonance in the transverse wave propagation (both perpendicular and parallel to the magnetic field). For spin-orbit mixed states, one finds zeros of the longitudinal dielectric constant (for long wavelength) near the electron spin-flip frequency. The mechanism for the spin wave associated with this zero is a correlation of the motion of electrons with "opposite spins" by the long-range Coulomb field through the spin-orbit coupling of the crystalline eigenstates.

I. INTRODUCTION

Spin-wave excitations of conduction electrons in solids are usually discussed in relation to the properties of itinerant ferromagnets,[1] and of simple metals[2] (in an applied magnetic field) in which weak exchange[3] interactions are important. In both cases essential roles are played by exchange interactions and the magnetization of the equilibrium state, ferromagnetism in the former and conduction electron paramagnetism in the latter. In this paper we show that interesting spin-wave effects occur in a solid-state plasma for which simple self-consistent-field theory is appropriate and exchange interactions are unimportant.

It was pointed out in a preliminary publication[4] that, even when explicit exchange interactions are unimportant, electron spin waves can occur in nonmagnetic conductors owing to spin-orbit coupling and the Coulomb self-consistent field (SCF) of the interacting electrons. In this previous work, the collective excitations of the electron gas were treated in the longitudinal wave approximation. It was shown that the general SCF longitudinal dielectric constant had zeros (for long wavelength) near the electron spin-flip frequency. The mechanism for the spin wave associated with this zero is a correlation of the motion of electrons with "opposite spins" by the long-range Coulomb field through the spin-orbit coupling of the crystalline eigenstates. This paper gives a more general treatment of the SCF collective excitations of the quantum plasma in a magnetic field. A general implicit dispersion relation is derived by solving self-consistently the linearized equation of motion for the single-electron density matrix and the full set of Maxwell's equations. Exchange effects such as those encountered in an extended random-phase